AFESD - TDR - 63- 80

ABSTRACT

A receiver which employs coherent, or synchronous, detection must have a priori knowledge of the phase of the received signal. The receiver acquires this knowledge by performing measurements on the channel. The result of the measurement process is a noisy phase reference which is used by the receiver in the detection processing of the incoming signals. In this report, the effect of using baud decisions to direct the phase measurement process is investigated by means of computer simulation of a coherent communication system employing either orthogonal or phase-reversal signaling. Error rates are given for several signal-to-noise ratios. A major conclusion of this study is that phase measurements obtained through such a decision directed technique result in system error rates which are generally less than error rates of corresponding non-decision directed phase measurement schemes at all signal-to-noise ratios; no threshold exists below which a decision directed phase measurement system deteriorates rapidly.

I. INTRODUCTION

For a wide class of communication channels a lower error rate can be realized at a given received signal-to-noise ratio if the receiver employs coherent, or synchronous, detection rather than incoherent, or envelope, detection. To perform coherent detection, however, the receiver must know a priori the phase of the received signal; in general this knowledge must be obtained by measurements on the channel, either by using a "pilot tone" or by observation of the previous signals themselves.

In a recent study concerning the optimum division of transmitter power between pilot tone and information bearing signals, Van Trees[1] has shown that best over-all performance is achieved if the transmitter power is devoted entirely to the information bearing signals. We are therefore interested in optimum means for performing a channel measurement based on these signals.

Since the signals to be used in determining the channel phase are corrupted by noise, the measurement yields only an estimate of the true phase of the received signals. In a recent report Price[2] analyzed the effect of the "goodness" of this estimate on the performance of a binary communication system employing coherent detection with a particular form of channel measurement. By simulation on a digital computer we have studied empirically the performance of systems employing other channel measurement techniques with orthogonal as well as anticorrelated signals; in particular we have investigated the effect of using the baud decisions to direct the measurement process in order to obtain a less noisy measurement.

II. CHANNEL MEASUREMENT

The binary communication system under consideration consists of two signaling waveforms, say $s_1(t)$ and $s_2(t)$, which are of duration T (baud length) and are either orthogonal or anticorrelated, i.e.

$$\int_0^T s_1(t)\, s_2(t)\, dt = \begin{cases} 0, & \text{if orthogonal} \\ -E, & \text{if anticorrelated} \end{cases}$$

where

$$\int_0^T s_1^2(t)\,dt = \int_0^T s_2^2(t)\,dt = E$$

The disturbance is assumed to be additive, white gaussian noise.

The parameter which represents the length of the channel measurement is called the "effective measurement time," T_m. If the channel is time-invariant, T_m together with the received signal-to-noise ratio E/N_o provide a measure of the quality of the measurement; for a given E/N_o, knowledge of the received signal phase is improved as T_m is increased.

We shall express the effective measurement time T_m in units of the baud duration T, i.e., $T_m = \gamma T$ where (usually not but necessarily) $\gamma \geq 1$. If $\gamma = 1$, the channel measurement corresponds to the duration of a single baud and is usually carried out during the baud just preceding that for which a decision is to be made. This holds for both orthogonal and phase reversal signaling. For orthogonal signaling only, $\gamma = 1$ may also correspond to envelope detection, where the channel measurement is now derived from the incoming signal instead of the signal from the preceding baud. For extended measurement into the past, i.e., $\gamma > 1$, one may choose various forms of weighting of past inputs. If the channel is time-invariant, the type of weighting used will not be critical. However, if the channel phase shift is changing with time, one should form the channel measurement by choosing the weighting to match the channel characteristics.

As an example of the way in which channel measurement may be performed, let us suppose that the signals are orthogonal and the receiver consists of two matched filters, one matched to the "mark" waveform and the second matched to the "space" waveform. Furthermore, assume that $\gamma = 1$ so that the channel measurement is of duration T, and that this measurement is made in the baud just preceding the one on which a decision is to be made. The measurement can be made by passing the received signal into the pair of matched filters and sampling the outputs of these filters near the end of the measurement interval. If there were no additive noise, only one filter would have an output at the sampling instant. The amplitude and phase of the sum of the filter outputs would then be an exact measure of the gain and phase shift of the channel.

In the presence of additive noise, however, both filters will have a non-zero output at the sampling instant. At this point, a channel measurement can be obtained in either of two ways. The outputs of the two filters may be added to form an estimate of the channel signal strength and phase shift. This procedure, which we call non-decision directed channel measurement, causes the channel measurement to be degraded by the noise from both matched filters. Alternatively, one can first decide which of the two filters contains the signal and use this filter output as the reference for the detection of the signal during the next baud. This latter scheme will be called decision directed measurement.

Channel measurement in a matched filter system with orthogonal signaling may be extended to include more than one presignaling interval in the following manner. The two matched filters are sampled at the end of each baud, and a single sample is formed through either decision directed measurement or addition of the two filter outputs (non-decision directed measurement). This sample is then added to the weighted sum of such samples from earlier presignaling intervals. The weighting is with respect to the past and, as we previously mentioned, should be chosen to match the channel.

If the signaling waveforms are anticorrelated, as in differential phase-reversal signaling, channel measurement is performed in a different manner. Here, there is only a single matched filter output at the end of each baud. The amplitude of the noise-free filter output is proportional to the signal strength, but, since information is conveyed in phase differences between successive bauds, its phase is a function of the transmitted information as well as the channel phase shift. For this type of signaling we are constrained to use decision directed measurement in order to remove the modulation. If $\gamma = 1$, there is no problem, as we are using the measurement from the previous baud as the phase reference for the detection of the incoming signal. However, for $\gamma > 1$, the channel measurement is formed by a weighted addition of more than one baud, requiring that the information carrying phase modulation be removed from the individual bauds prior to their addition. The modulation is removed from each phase measurement by reversing its phase if it differs by more than $\pm 90^\circ$ from the reference phase; otherwise, the phase measurement is left unaltered.

For both orthogonal and anticorrelated signaling, decision errors will occur in the presence of additive noise. It is not clear _a priori_ which of the two channel

measurement techniques will result in fewer decision errors. In a practical system we have a choice of the type of channel measurement technique to use if the signals are orthogonal. For anticorrelated signals, this choice no longer exists.

An analytical comparison between the channel measurement techniques has not proved mathematically tractable. As an alternative, an orthogonal and a differential phase-reversal signaling system were simulated on a digital computer; for each the error rates with decision and non-decision directed channel measurement were compared. In the phase-reversal signaling system, we simulated an artificial non-decision directed technique to form a basis for comparison with the decision directed channel measurement. *

The results of the simulation are presented in following sections.

III. ORTHOGONAL SIGNALING

A matched filter detection system with orthogonal signaling was simulated, using decision and non-decision directed channel measurement with both exponential and uniform weighting into the past. The simulation used a simple detection procedure. Two noise vectors were formed from a sample of normally distributed zero-mean and unit variance pseudo-random numbers (Appendix A), and a signal component was added to one of the noise vectors. If $\vec{V}_1$ contains the signal, then $\vec{V}_1 = (X_1 + s + jY_1)$ and $\vec{V}_2 = (X_2 + jY_2)$, where X_1, Y_1, X_2 and Y_2 are mutually independent, zero-mean and unit variance normally distributed variables.† Using these two vectors, dot products were formed with the reference vector resulting from the channel measurement. The two dot products (corresponding to the outputs of two matched filters) were compared and the largest chosen as containing the signal component.

When decision directed channel measurement was used, the output of the filter which gave the largest dot product was added to the weighted sum of the previous measurements. For non-decision directed measurement, the two filter outputs were summed and added to the weighted sum of previous measurements. For each scheme the number of errors in 5000 trials are shown in Table 1.

The value of γ indicated in this table corresponds to the length of channel measurement. For each value of γ the first row contains the results for decision

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* As explained later, we assumed that the "measurement portion" of the receiver knew exactly the transmitted message.

† It follows that $E/N_o = 1/2\ s^2$.

TABLE I

ORTHOGONAL SIGNALING SYSTEM

Uniform Weighting

γ \ E/N_0	.5	1.0	1.5	2.0	3.0	4.5	8.0
1	1867 1963 1992	1352 1550 1559	1011 1208 1241	729 963 999	383 599 605	144 293 291	20 49 57
2	1612 1801	1074 1289	738 906	526 672	251 345	115 150	17 20
5	1370 1503	874 997	616 680	448 488	244 265	99 116	14 15
10	1263 1370	828 865	597 595	431 436	228 249	102 106	15 15

Exponential Weighting

1	1867 1963	1352 1550	1011 1208	729 963	383 599	144 293	20 49
2	1614 1782	1108 1287	727 910	517 684	264 364	109 153	17 18
5	1380 1533	867 1007	613 684	440 475	236 260	105 110	15 15
10	1264 1368	833 882	595 604	431 448	228 242	99 106	14 13

Number of Errors in 5000 Trials
First row -- decision directed measurement
Second row -- non-decision directed measurement
Third row -- envelope detection ($\gamma = 1$ only)

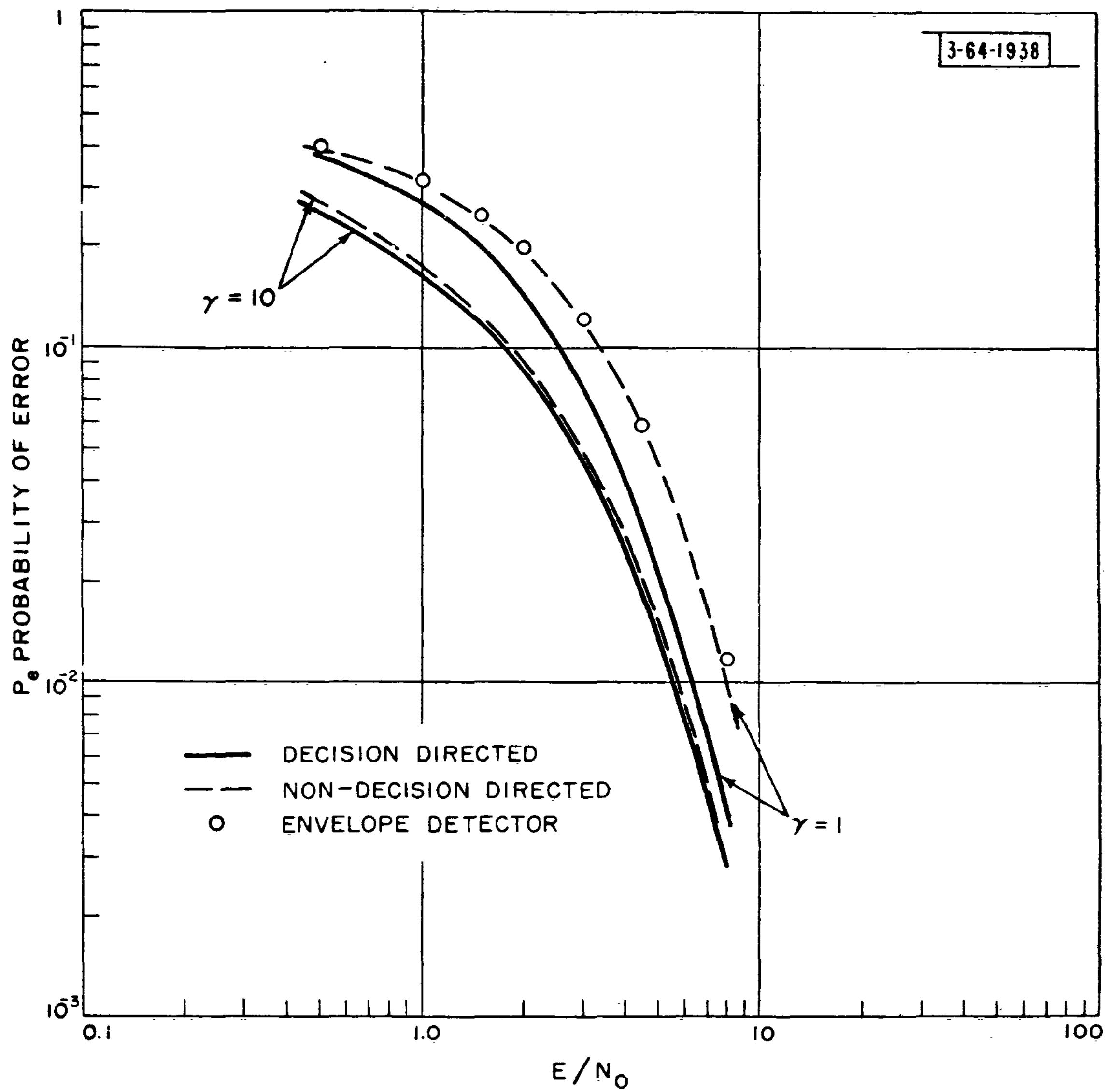

Fig. 1 Probability of Error for Orthogonal Signaling with Exponential Weighting

directed measurement, the second for non-decision directed measurement. For the special case of uniform weighting with $\gamma = 1$, two possibilities exist for non-decision directed measurement: either the previous baud or the present baud may be used as reference, the latter case corresponding to envelope detection. The results for these two alternatives are shown in the second and third lines, respectively; they are quite similar, as might be expected from the analytical comparison presented in Appendix B.

For exponential weighting, it can be shown that the equivalent value of γ is given by the relation

$$\gamma = \frac{1 + D}{1 - D}$$

where D is the decrement ($D < 1$) used in the channel measurement to provide weighting into the past. (At the end of each signaling interval, the new reference vector for the incoming signal is formed by adding the measurement from the present signaling interval to D times the previous reference vector, i.e., $\vec{V}_R = \vec{V}_0 + D\vec{V}_{R-1} = \vec{V}_0 + \sum_{i=1}^{\infty} D^i \vec{V}_{-i}$). The values of $D = 0$, $1/3$, $2/3$ and $9/11$ were used, corresponding to $\gamma = 1, 2, 5$, and 10.

The improvement in system performance resulting from the use of decision directed measurement is evident from Figure 1. As might be expected this improvement is much more pronounced for small values of γ than for the large values at which the measurement signal-to-noise ratio is already good. It is especially interesting to note that even for very low E/N_0 the corresponding high error rate does not degrade the decision directed measurement; no threshold exists, below which the decision directed system deteriorates rapidly.

IV. DIFFERENTIAL PHASE-REVERSAL SIGNALING SYSTEM

In this section we present results from the simulation of four differential phase-reversal systems. In each, information is conveyed by means of reversals of phase between successive bauds. The systems differ only in the method of channel measurement by which a phase reference is obtained. Coherent detection was performed by forming the dot product between the incoming signal and the phase reference vector; the decision as to which symbol was transmitted is determined by the sign of this dot product.

Since information is conveyed by phase reversals between successive bauds, an error in any baud will very likely be followed by an error in the succeeding baud, i.e.,errors will generally occur in pairs.

As a reference (but physically unrealizable) phase-reversal signaling system, we assumed that the measurement portion of the receiver knew exactly the transmitted message; the receiver then formed the reference vector by multiplying the received signal-plus-noise vector for each baud by either plus or minus one depending on the (known) sign of the transmitted phase. This type of measurement is really a case of non-decision directed channel measurement using phase-reversal signaling. Although physically unrealizable, it has two advantages as a basis for comparison; error rates can be predicted analytically,[2] and the difference between the error rate for this technique and for decision directed measurement is attributable directly to the effect of errors on the phase reference.

As a modification of the system described above, we consider a system in which bauds in error are not added into the measurement. The channel measurement is now decision directed and, for all practical purposes, is still physically unrealizable.* Note that the manner in which the reference vector is obtained rules out uniform weighting of a small number of past measurements, since a number of successive errors will cause the loss of the reference vector.

The third system employed decision directed channel measurements and is physically realizable. The measurement was performed by multiplying the incoming signal-plus-noise vector for each baud by minus one if this vector differed from the reference vector by more than $\pm 90^{\circ}$, i.e., the sign of the dot product was negative; otherwise, the incoming vector was left unaltered. With the modulation thus removed (if the baud decision was correct), the signal-plus-noise vector for each baud was added to the weighted past measurements.

In the fourth system the phase modulation was removed from the channel measurement by squaring the signal-plus-noise vector received during each baud. The squared vector was then added to the weighted sum of squared vectors from previous bauds to

* At least in principle,either of the above techniques could be realized with error-correcting coding.

form a reference vector having (in the absence of noise) twice the true reference phase. The phase reference desired for the detection processing of the incoming signal was obtained by dividing the phase of this reference vector by two. As a result of the squaring operation, there is a loss in the channel measurement signal-to-noise ratio.

For each of these four systems the number of errors occurring in 10,000 trials is listed in Table 2 (also, see Fig. 2). The weighting is exponential in all cases. For each value of γ, the error rates are given in the order in which the systems were discussed, i.e.

(a) First row--non-decision directed channel measurement

(b) Second row--errors omitted from channel measurement

(c) Third row--decision directed channel measurement

(d) Fourth row--square and divide channel measurement

For $\gamma = 1$, the reference vector used in the detection processing of the incoming signal is the signal received in the previous baud. For this case, the best non-decision directed measurement scheme can perform as well but not better than its decision directed counterpart. Hence, at best, the two channel measurement techniques result in an identical error rate. In the second system, the manner in which the reference vector was obtained precluded any results for the $\gamma = 1$ case.

For $\gamma = 2$ and all E/N_0, the error rate of the receiver employing decision directed channel measurement was (statistically) significantly lower than that obtained from the receiver which used non-decision directed measurement. For all practical purposes, the performances of the two measurement schemes for both $\gamma = 5$ and for $\gamma = 10$ are identical. Observation of results from the second row (for all γ) indicate that very little is gained by omitting errors from the measurement.

Perhaps the most interesting comparison is that between the third and fourth rows of Table 2 (also, see Fig. 2), i.e.,the two realizable systems. Especially for short measurement times (small γ) decision directed measurement results in significantly (Appendix C) fewer errors. In view of the relatively small difference in performance between the two measurement techniques, it is questionable that one would exchange the square-law measurement scheme for the more complicated (circuit-wise) decision directed procedure.

TABLE 2

DIFFERENTIAL PHASE-REVERSAL SIGNALING SYSTEM

Exponential Weighting

γ \ E/N_0	.5	1.0	1.5	2	3	4.5
1	---- ---- 3141 3494	---- ---- 1922 2333	---- ---- 1217 1502	---- ---- 750 975	--- --- 291 428	--- --- 66 104
2	3387 3413 3154 3350	2085 2091 1887 2093	1199 1203 1070 1237	697 701 629 733	246 248 221 250	44 42 39 46
5	3171 3081 3131 3221	1789 1761 1786 1867	1011 995 1014 1047	552 555 557 586	192 192 191 192	36 36 37 30
10	3027 2979 3047 3135	1647 1631 1667 1739	929 931 946 943	534 524 529 550	176 174 173 178	30 30 31 30

Number of Errors in 10,000 Trials

First row --non-decision directed channel measurement
Second row -- errors omitted from channel measurement
Third row -- decision directed channel measurement
Fourth row -- square and divide channel measurement

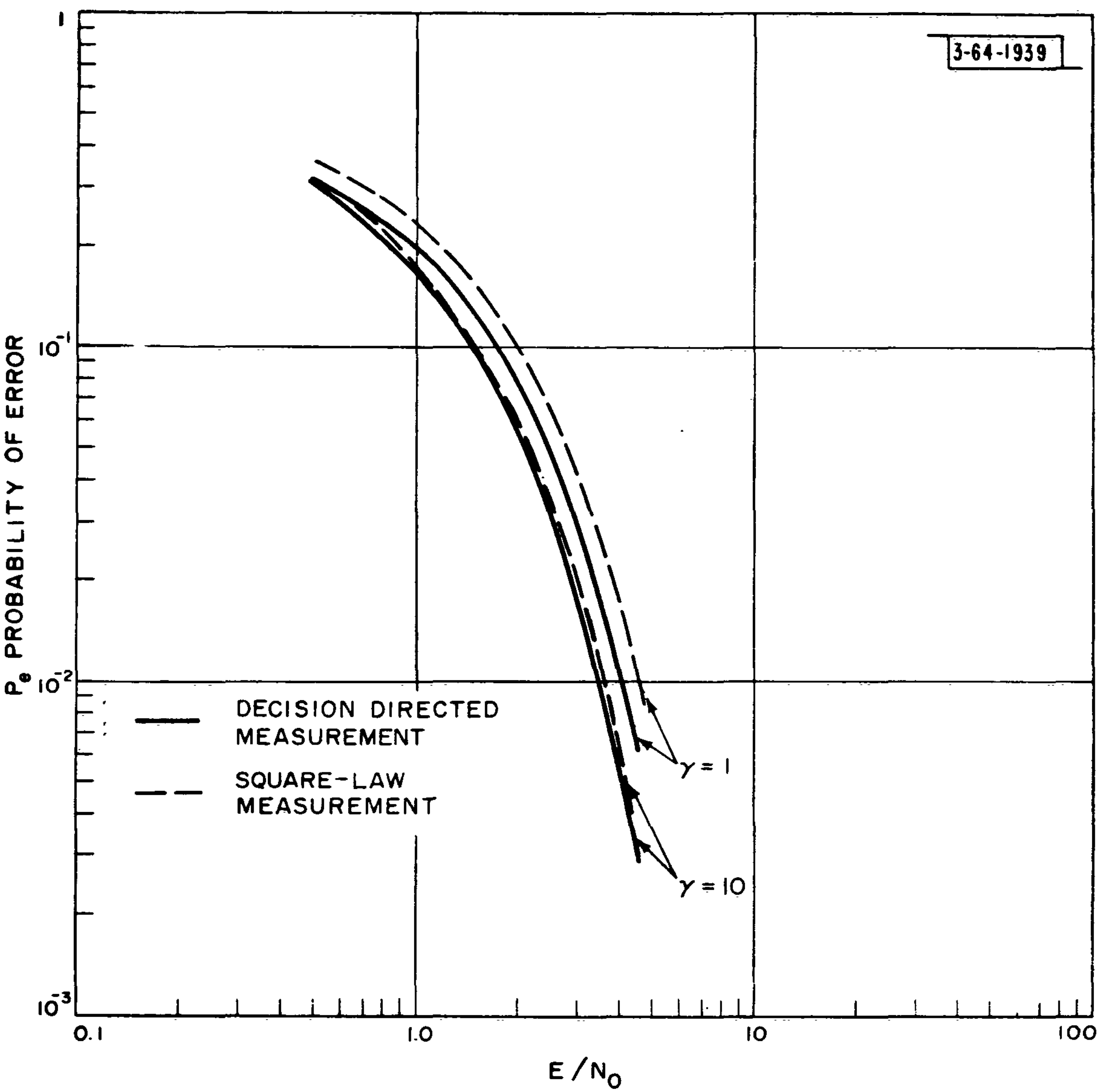

Fig. 2 Probability of Error for Differential Phase-Reversal Signaling with Exponential Weighting

In conclusion, it is interesting to note again, that even at low E/N_0 the corresponding error rate does not degrade the decision directed measurement system.

V. SYSTEM "START-UP"

In the tabulated results of the previous sections, the initial phase reference was a noise vector chosen from the set of random vectors. This procedure of turning on the receiver is a reasonable one. In practice, receiver noise will always be present and will have the same effect.

Through simulation, the problem of initial acquisition of a reference vector was studied in more detail. We assume that the transmitter had been turned off and that the receiver had been operating only on noise for a period of time long compared to T_m. Then the transmitter was turned on and we examined the baud to baud buildup of the reference vector.

For non-decision directed measurement, one would expect the receiver to establish a meaningful phase reference in a period of time comparable with the effective measurement time. Our simulation results substantiated this conjecture. In fact for high signal-to-noise ratios, initial acquisition was obtained in a time less than T_m.

In the case of decision directed channel measurement it is difficult to predict how rapidly the receiver will establish a good reference. Our simulation results, which were obtained for exponential weighting with $\gamma = 10$ and E/N_0 between 0.5 and 8.0, indicate the following conclusions: (a) for small measurement times and all signal-to-noise ratios, the receiver acquires a proper measurement rapidly; (b) for long measurement times, as for example $\gamma = 10$, and large E/N_0, the receiver acquires a good measurement in a time comparable to (and sometimes less than) T_m; (c) for long measurement times and small E/N_0, the receiver acquires a good measurement in a period of time which is usually greater than T_m but always less than $2T_m$.

From the above we may conclude that a receiver need not be primed when operating with either decision directed or non-decision directed channel measurement.

APPENDIX A

Generation and Characteristics of the Random Number Sample

The 20,000 normally distributed, zero mean and unit variance, pseudo-random numbers used in the simulation were generated on the computer.* This was done by first generating a number of period 2^{23}, and uniformly distributed in the interval (0, 1). The computation procedure is as follows:

a) Compute $r_i = Mr_{i-1}(\text{Mod } 2^{35})$, $i = 1, 2, \ldots$

b) Designating the floating point equivalent of r_i by u_i, compute

$$v_i = \sqrt{-2 \operatorname{Log}_e [.5(1-|1-2u_i|)]}$$

c) Then,

$$N_i = m + \sigma \left\{ \operatorname{sign}(u_i - .5) \left[v_i - \frac{a_0 + a_1 v_i + a_2 v_i^2}{1 + b_1 v_i + b_2 v_i^2 + b_3 v_i^3} \right] \right\}$$

where N_i is the normal variable of mean m and variance σ^2, and r_0 = initial random number as a fixed point binary odd integer ($r_0 = 2^{36} - 1$ has been used to start the routine).

$M = 5^{13}$

$a_0 = 2.515577$	$b_1 = 1.432788$
$a_1 = .802853$	$b_2 = .189269$
$a_2 = .010328$	$b_3 = .001308$

The total time to generate a single number is approximately 12 milliseconds.

Some of the characteristics of the pseudo-random sample were investigated. The 20,000 numbers were divided into two sets, the X-sample being comprised of the first set of 10,000 numbers and the Y-sample being comprised of the second set of 10,000 numbers. The cdf of each set is plotted in Figure 1A. As can be observed, the Y-sample is a better fit to the normal distribution than the X-sample. Application of the χ^2 goodness of fit test (Tables A1 and A2) verifies this same conclusion.

* Routine AANDRN

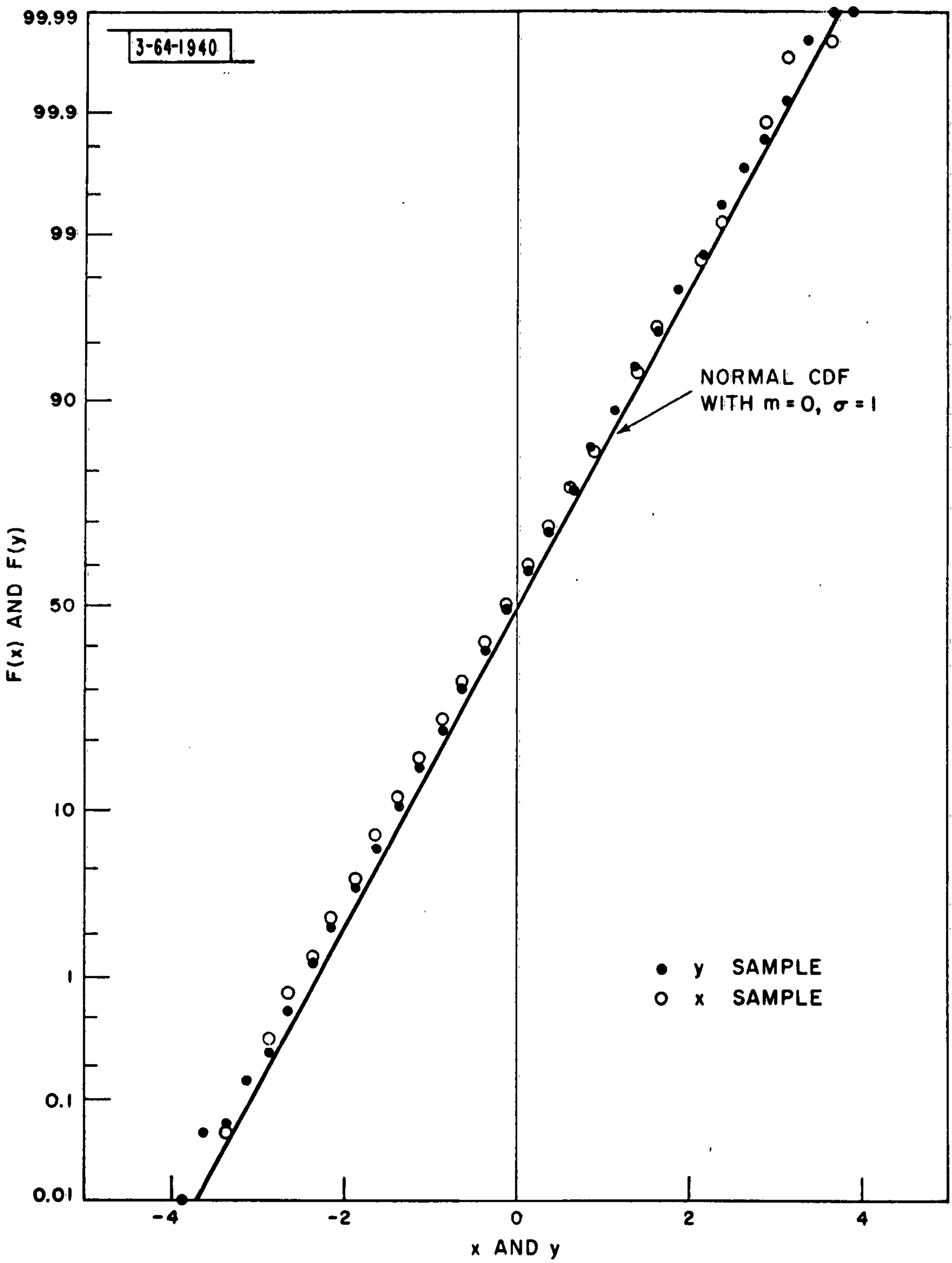

Fig. 1A Cumulative Distribution Function of the Two Sets of 10,000 Pseudo-random Numbers

TABLE A1

Range Box	ν_i No. of Samples		p_i	np_i	$\frac{(\nu_i - np_i)^2}{np_i}$
-100.00000	0				
-4.00000	0				
-3.75000	0	15	.00135	13.5	.17
-3.50000	5				
-3.25000	10				
-3.00000	18		.00165	16.5	.14
-2.75000	44		.0032	32.	4.51
-2.50000	66		.0060	60.	.60
-2.25000	115		.0105	105.	.95
-2.00000	168		.0173	173	.14
-1.75000	301		.0267	267	4.33
-1.50000	416		.0388	388	2.02
-1.25000	548		.0530	530	.61
-1.00000	691		.0680	680	.18
-0.75000	797		.0819	819	.59
-0.50000	893		.0927	927	1.25
-0.25000	974		.0988	988	.20
0.25000	956		.0988	988	1.04
0.50000	924		.0927	927	0.
0.75000	799		.0819	819	.49
1.00000	630		.0680	680	3.68
1.25000	538		.0530	530	.12
1.50000	405		.0388	388	.74
1.75000	285		.0267	267	1.21
2.00000	172		.0173	173	0.
2.25000	100		.0105	105	.25
2.50000	64		.0060	60	.27
2.75000	52		.0032	32	12.50
3.00000	17		.00165	16.5	.02
3.25000	9				
3.50000	1				
3.75000	0	12	.00135	13.5	.17
4.00000	1				
100.00000	1				
					36.17

$$P_r(\chi^2_{25} > 36.17) = .07$$

TABLE A2

Range Box	ν_i No. of Samples	p_i	np_i	$\frac{(\nu_i \; np_i)^2}{np_i}$
-100.00000	1			
-4.00000	0			
-3.75000	4 } 14	.00135	13.5	.02
-3.50000	1			
-3.25000	9			
-3.00000	10	.00165	16.5	2.56
-2.75000	32	.0032	32.	0.
-2.50000	73	.0060	60.	2.82
-2.25000	97	.0105	105.	.61
-2.00000	167	.0173	173.	.21
-1.75000	257	.0267	267.	.37
-1.50000	375	.0388	388.	.44
-1.25000	552	.0530	530.	.91
-1.00000	690	.0680	680.	.15
-0.75000	799	.0819	819.	.49
-0.50000	914	.0927	927.	.18
-0.25000	962	.0988	988.	.68
0.25000	960	.0927	927.	.01
0.50000	930	.0819	819.	2.82
0.75000	867	.0680	680.	1.41
1.00000	711	.0530	530.	.05
1.25000	525	.0388	388.	.44
1.50000	401	.0267	267.	2.16
1.75000	243	.0173	173.	.47
2.00000	182	.0105	105.	0.
2.25000	105	.0060	60.	2.00
2.50000	71	.0032	32.	.28
2.75000	29	.00165	16.5	.02
3.00000	16			
3.25000	9			
3.50000	6			
3.75000	1 } 17	.00135	13.5	.91
4.00000	0			
100.00000	1			
				20.80

$$P_r\,(\chi^2_{25} > 20.8) = .70$$

To test the independence of the samples, the autocorrelation function of the set of 20,000 random numbers was computed for shifts through forty and is displayed in normalized form in Figure 2A.

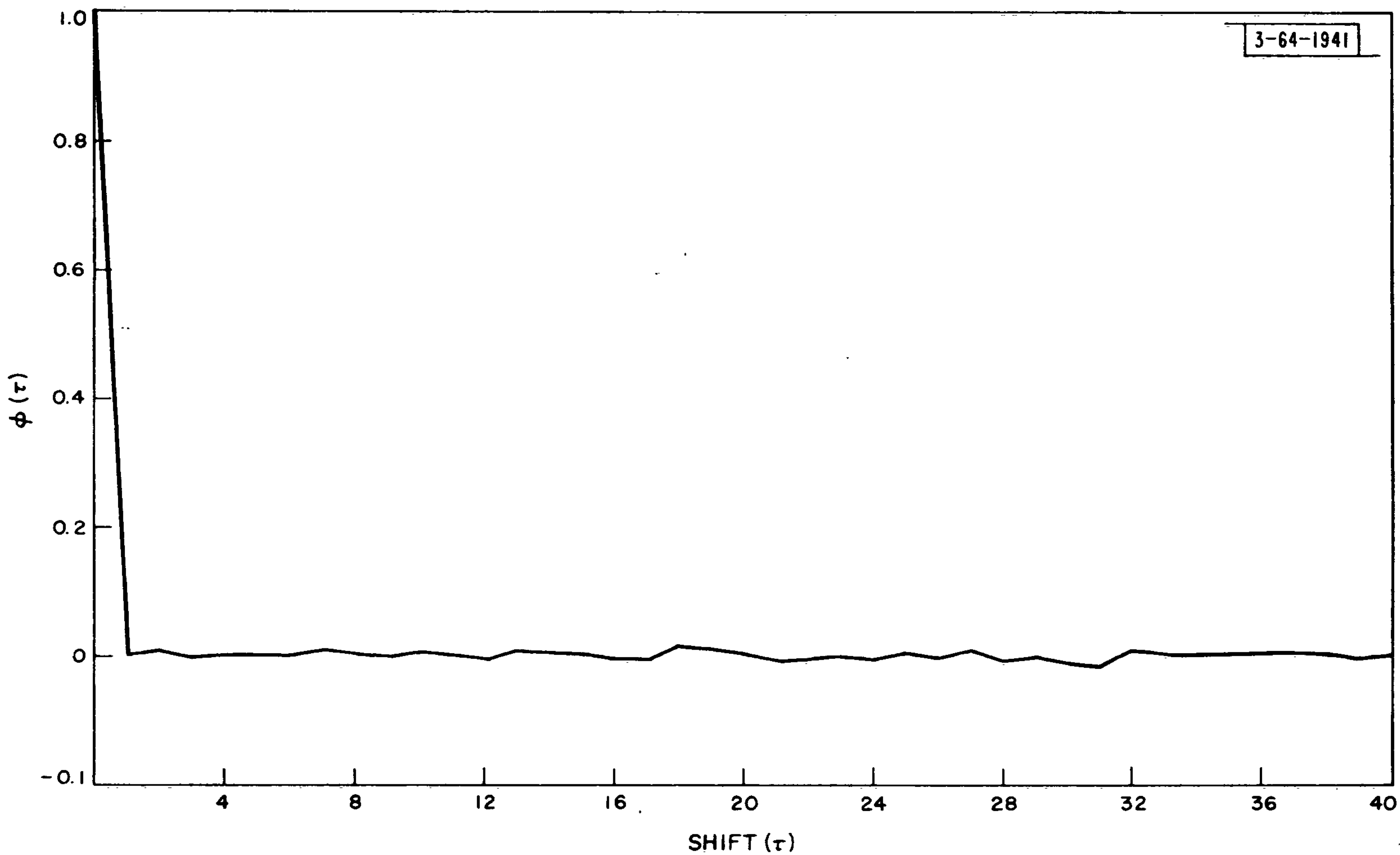

Fig. 2A Autocorrelation Function of the 20,000 Pseudo-random Numbers

APPENDIX B

Statistical Properties of Dot Product Differences in Detection of Orthogonal Signals

(a) Envelope detection vs. non-decision directed measurement with $\gamma = 1$.

For envelope detection, the decision quantity (DQ) is just

$$DQ = (X_1 + s)^2 + Y_1^2 - X_2^2 - Y_2^2 \tag{B-1}$$

where s is the signal voltage, $(X_1 + jY_1)$ is the additive noise vector from the matched filter containing the signal, and $(X_2 + jY_2)$ is the additive noise vector in the second matched filter. The X_i and Y_i are assumed to be independent, zero-mean and unit variance normal variables. Then,

$$E(DQ) = s^2 \tag{B-2}$$

and

$$\begin{aligned} E\{(DQ)^2\} &= E\{(s^2 + 2sX_1 + X_1^2 + Y_1^2 - X_2^2 - Y_2^2)^2\} \\ &= 8\sigma^4 + 4s^2\sigma^2 + s^4 \\ &= 8 + 4s^2 + s^4 \end{aligned} \tag{B-3}$$

This gives a variance for (DQ) of $8 + 4s^2$.

In the case of non-decision directed measurement where only the previous baud is used as the channel measurement for the incoming signal, the (DQ) is

$$(DQ) = (s + X_1 + X_2)(s + X_3 - X_4) + (Y_1 + Y_2)(Y_3 - Y_4) \tag{B-4}$$

Again,

$$E(DQ) = s^2 \tag{B-5}$$

and

$$E\{(DQ)^2\} = E\{s + X_1 + X_2)^2 (s + X_3 - X_4)^2 + (Y_1 + Y_2)^2 (Y_3 - Y_4)^2\}$$

$$= 8\sigma^4 + 4s^2\sigma^2 + s^4 \tag{B-6}$$

Therefore,

$$\text{var }(DQ) = 8 + 4s^2 \tag{B-7}$$

The means and variances of the decision quantity in the two schemes are identical. In view of this, the expected error rate from the two measurement techniques should be similar.*

(b) Decision directed vs non-decision directed channel measurement with $\gamma = 1$.

For convenience, let us choose uniform weighting with $\gamma = 1$ and examine the mean and variance of the decision quantity for the two types of channel measurement. In the case of non-decision directed measurement, the mean and variance of (DQ) are given by Equations (B-5) and (B-7).

In the case of decision directed measurement, let us assume that the reference vector consists of signal plus noise, i.e., the previous baud was received correctly. Then,

$$DQ = \vec{V}_R \cdot (\vec{V}_{s+n} - \vec{V}_n)$$

$$= (X_R + s)(X_1 - X_2 + s) + Y_R(Y_1 - Y_2) \tag{B-8}$$

The mean of (DQ) is s^2. The second moment is

$$E\{(DQ)^2\} = 4\sigma^4 + 3s^2\sigma^2 + s^4$$

$$= 4 + 3s^2 + s^4 \tag{B-9}$$

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* If the distribution functions of (DQ) in the two measurement schemes were identical, the expected error rates should also be identical.

and

$$\text{var (DQ)} = 4 + 3s^2 \qquad \text{(B-10)}$$

if $\vec{V}_R$ is the correct vector.

Suppose now that the reference vector is pure noise so that

$$\vec{V}_R = X_R + jY_R$$

Then,

$$DQ = X_R\,(X_1 - X_2 + s) + Y_R(Y_1 - Y_2) \qquad \text{(B-11)}$$

and

$$\mathbf{E}(DQ) = 0 \qquad \text{(B-12)}$$

Also,

$$\text{var (DQ)} = \mathbf{E}\{(DQ)^2\} = \mathbf{E}\{X_R^2(X_1^2 + X_2^2 + s^2) + Y_R^2(Y_1^2 + Y_2^2)\}$$

$$\text{var (DQ)} = 4\sigma^4 + s^2\,\sigma^2$$

$$\text{var (DQ)} = 4 + s^2 \qquad \text{(B-13)}$$

When the system is operating at low error rates, the decision directed scheme is clearly better than non-decision directed measurement because of the difference in the variance of (DQ). For high error rates, decision directed measurement continues to yield a smaller variance for (DQ) but, on the average, the mean of (DQ) is also decreased.

APPENDIX C

Variance of Error Rate Estimates

The variance of the error rate estimates given in the tables is difficult to calculate for the decision directed measurement techniques because of the dependence of errors between bauds. However, for orthogonal signaling and non-decision directed measurement, the errors between bauds are independent. In this case, the standard deviation about the expected value of the error rate is

$$\sigma = \sqrt{\frac{P_e(1-P_e)}{n}}$$

where P_e is the expected error rate, i.e. the true error rate, and n is the total number of samples. For 5000 trials, and $P_e = .3$ we obtain $\sigma = 32.5$ errors; if $P_e = .01$, $\sigma = 7$ errors. For 10,000 trials, and $P_e = .3$ we obtain a $\sigma = 46$ errors; if $P_e = .01$, $\sigma = 10$ errors.

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